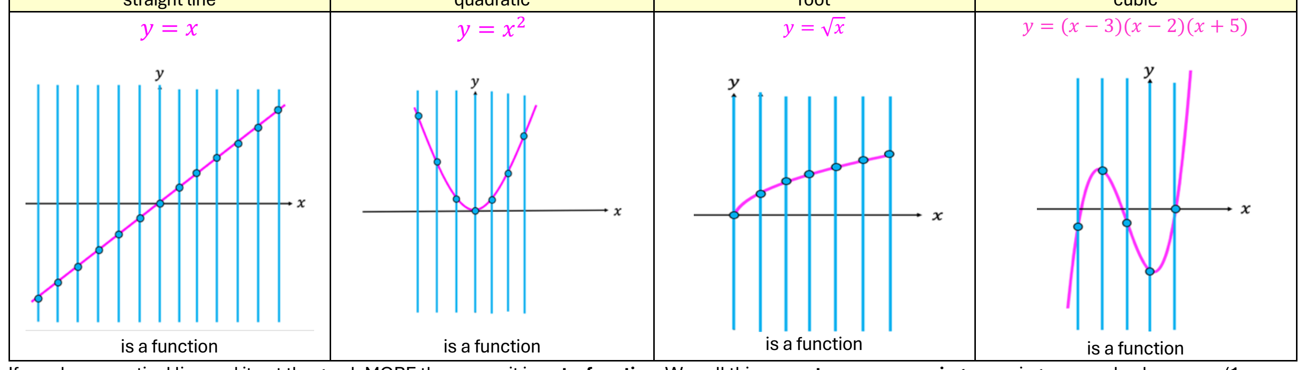
Functions Versus Mappings

To be a function the GRAPH must **pass the vertical line test**. Any vertical line drawn MUST only cut the graph ONCE (we need none of the lines to cross the graph more than once)

Examples of functions

straight line	quadratic	root	cubic
$y = x$	$y = x^2$	$y = \sqrt{x}$	$y = (x-3)(x-2)(x+5)$

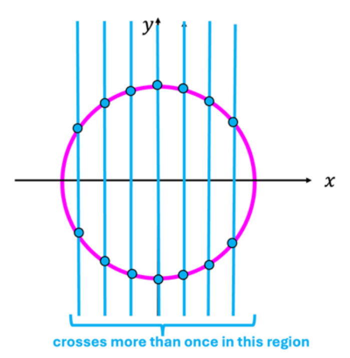


If we draw a vertical line and it cut the graph MORE than once, it is **not a function**. We call this a **one-to-many mapping**, meaning one x value has many (1 or more) y values.

Example of NOT A function (known as a one to many mapping/relation)

circle

$x^2 + y^2 = 4$



To be a function the graph must pass the vertical line test. If not a function we say a one to many relation/mapping

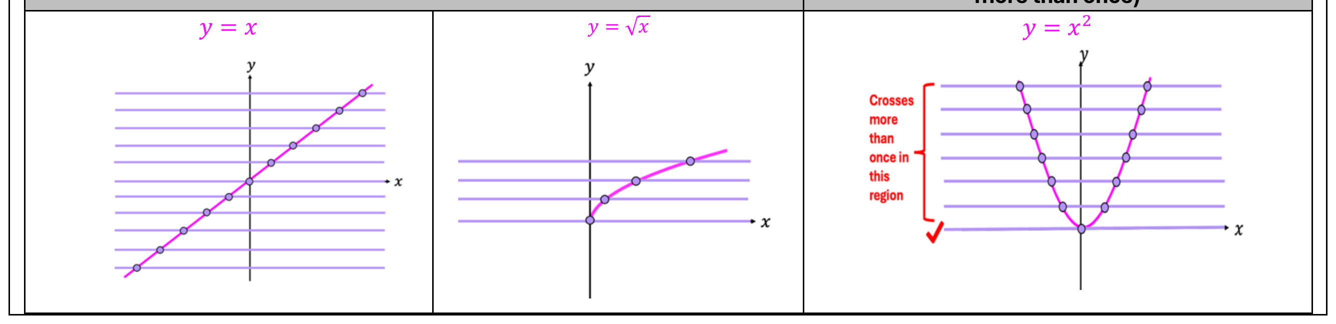
not a function!

When something is not a function we call it a one-to-many mapping.

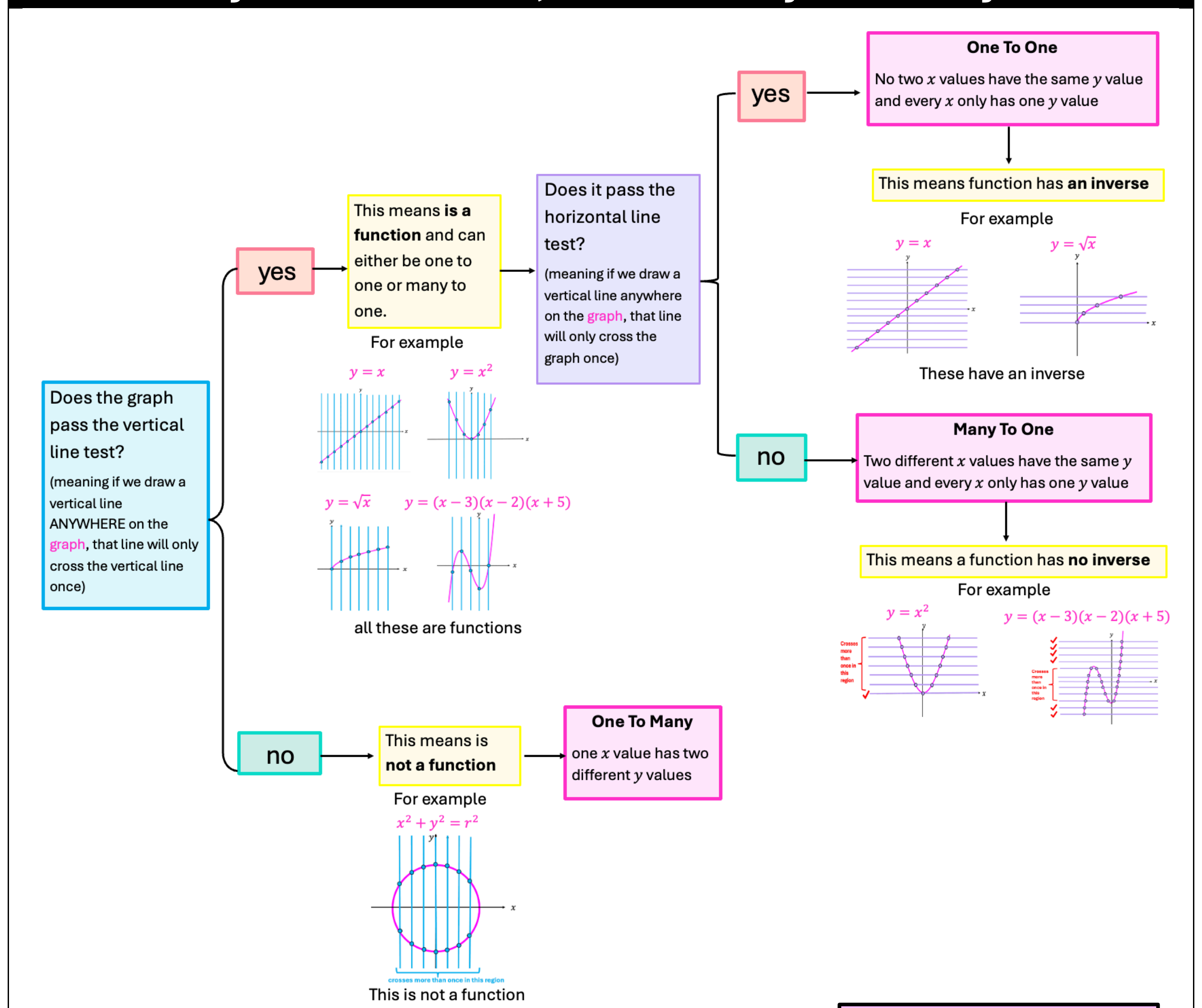
Types Of Functions – One to One Versus Many To One

So, we can tell when something is either a function or a one-to-many mapping. If something is a function, it can either be one-to-one or many-to-one.

One To One Function (draw a horizontal line, must only cross the graph once)	Many to One Function (draw a horizontal line, can cross the graph more than once)
$y = x$, $y = \sqrt{x}$	$y = x^2$



Summary Of One To One, One To Many and Many To One



Even though it comes up in the textbook, you don't need to know most of this for exam questions (it is too basic to be asked). The only thing you need to know is that when a function is many to one (doesn't have an inverse) it does not have an inverse. There is often a 1 marker saying why does a function have an inverse and it is because the function is many to one. You will see inverses in more detail later (how to find inverses which is the same as your GCSE knowledge)

Functions are either one to one or many to one. To have an inverse a graph must be one to one (pass the horizontal line test)

Basics (Recap from GCSEs)

Intuition

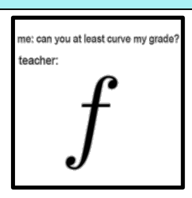
A function relates an input to an output. It is like a machine that has an input and an output.
For example, let's say the function tells us to multiply by 3 and subtract 4.

Input	Function Relationship/Rule	Output/Result
0	× 3 then − 4	$3(0) - 4 = -4$
1	× 3 then − 4	$3(1) - 4 = -1$
3	× 3 then − 4	$3(3) - 4 = 5$
8	× 3 then − 4	$3(8) - 4 = 20$
...	...	...
x	× 3 then − 4	$3(x) - 4 = 3x - 4$

Notation

First of all, it is useful to give a function a name.

The most common name is " f ", but we can have other names like " g " ... or even " dog. "



Notation:
" $f(x) = ...$ " is the most common choice way of writing a function (we use the letter f more than any other letter)

$$f(x) = 3x - 4$$

function name, input, relationship/rule

We read this as: " f of x " equals $3x - 4$

This means f takes x, multiplies it by 3 and then subtracts 4

Two important things to note:

- The most common name of a function is " f ", but there are also other commonly used names like " g " or " h ". They all mean the same thing as y. It is important to make sure you understand that we don't have to give a function a name, we can also just call it y.
$f(x) = x^2$
$g(x) = x^2$
$h(x) = x^2$
- We don't need to always use the letter " x " inside the bracket, it is just a place-holder, so we don't get too concerned about " x ", it is just there to show us where the input goes and what happens to it. It could be anything!
$f(x) = 1 - x^2 + x^3$
$f(a) = 1 - a^2 + a^3$
$f(☺) = 1 - ☺^2 + ☺^3$

Basics (Recap from GCSEs Continued)

Example 1 (very basic)
$f(x) = 3x - 4$. Find $f(5)$

Let's colour code to explain better

$$f(x) = 3x - 4$$

In English, $f(5)$ is saying what is the value of f when $x = 5$ which we can find by using a given rule

- Function f
- Input 5
- Relationship/rule is $3x - 4$

We **plug** the input 5 into the relationship/rule $3x - 4$ for the function f to find the output which is the solution to the question

$$f(5) = 3(5) - 4$$
$$= 11$$

Notice how we end up with an output/solution of 11

In summary, to find the value of the function at the point $x = 5$, we **plugged** the value of 5 into the relationship

A function just lates an input to an output by using its rule

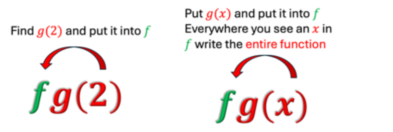
Reflection: You should now understand that function relates an input to an output – it's just a rule that takes an input of x and "spits" out a y value. Just think of a function as saying, give me an x value and I'll return a y value to you.

Example 2	Example 3	Example 4:	Example 5: With harder algebra
$f(x) = x^2 + 2x - 3$ i.Find $f(3)$ ii.Solve $f(x) = 5$	$f(x) = 5 + x$ Solve $f(a) = 7$	$g(x) = \frac{1}{x+3}$ Given that $g(a) = \frac{1}{10}$, find a	$f(x) = \frac{2x}{3x+5}, g(x) = \frac{3}{x+4}$. Solve $f(x) = g(x)$
$f(x) = x^2 + 2x - 3$ i. Find $f(3)$ Replace every x in the rule for the function with 3 $f(3) = 3^2 + 2(3) - 3$ $= 12$ ii. $f(x) = 5$ Here we are given the output and need to **work backwards** Replace $f(x)$ with it's relationship/rule $x^2 + 2x - 3 = 5$ $x^2 + 2x - 8 = 0$ $(x-2)(x+4) = 0$ $x = 2, -4$	$f(x) = 5 + x$ First work out what $f(a)$ is by putting a in place of x $f(a) = 5 + a$ We can now solve $f(a) = 7$ $5 + a = 7$ $a = -2$	$g(x) = \frac{1}{x+3}$ First work out what $g(a)$ is by putting a in place of x $g(a) = \frac{1}{a+3}$ We can now solve $g(a) = \frac{1}{10}$	$f(x) = \frac{2x}{3x+5}, g(x) = \frac{3}{x+4}$. I will colour code in a less detailed way now as you should understand the topic by now $f(x) = g(x)$ $\frac{2x}{3x+5} = \frac{3}{x+4}$ Multiply both sides by $(3x+5)$ and $(x+4)$ in order to get rid of the fractions $(x+4)(2x) = 3(3x+5)$ Then we expand the brackets and use your knowledge of solving quadratics $2x^2 + 8x = 9x + 15$ $2x^2 - x - 15 = 0$ $(2x+5)(x-3) = 0$ $x = -2.5, 37$

Example 6: With logarithms	Example 7: With logarithms	Example 8: With exponentials	Example 9: With exponentials
$f(x) = 4lnx^2$ Find $f(e^2)$	$f(x) = 5 + \ln\left(\frac{1}{x}\right)$ Find $f(e^2)$	$f(x) = 5e^{2x}$ Find $f(ln3)$	$f(x) = e^{-2x} - 3$ Find $f(ln3)$
$f(x) = 4\ln x^2$ $f(e^2) = 4\ln(e^2)^2 = 4\ln e^4 = 16\ln e = 16$	$f(x) = 5 + \ln\left(\frac{1}{x}\right)$ $f(e^2) = 5 + \ln\left(\frac{1}{e^2}\right)$ $= 5 + \ln e^{-2}$ $= 5 - 2\ln e$ $= 5 - 2$ $= 3$	$f(\ln 3) = 5e^{2\ln 3}$ $= 5e^{\ln 9}$ $= 5(9)$ $= 45$	$e^{-2\ln 3} - 3 = e^{\ln\frac{1}{9}} - 3$ $= \frac{1}{9} - 3$ $= -\frac{26}{9}$

Composite Functions

A composite function is a function applied to another function. This typically looks like $fg(2)$ or $gf(2)$ or $fg(x)$ or $gf(x)$. The latter two might seem scary since there are no numbers, but going through example 1 part ii. below will clear this up.



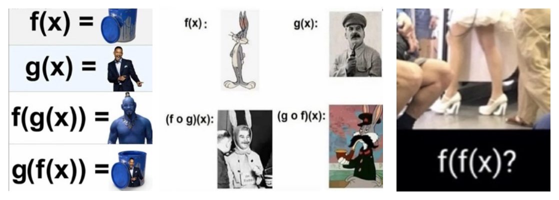
Always work from **right to left**

For composite functions we work from **right to left** (put the right inside function into the left outside function) i.e. $fg(x)$ means plug $g(x)$ into $f(x)$.
Following the examples will give you the best insight on this.

Example 1:	Example 2:	Example 3:	Example 4:
$f(x) = x^3 + 1, g(x) = x - 1$ Find i. $fg(2)$ ii. $fg(x)$ iii. $(f \circ f \circ f)(1)$	$g(x) = \frac{1}{x-2}, h(x) = \frac{2x+1}{3x+4}$ Find i. $gh(1)$ ii. $hg(x)$	$f(x) = \frac{2}{x}, g(x) = \frac{x+1}{x}$ Solve $gf(a) = 3$	$f(x) = 2x - 3, g(x) = x^2 + 2$ Solve $fg(x) = gf(x)$
i. $fg(2)$ Do inside function: $g(2) = 2 - 1 = 1$ Put this into the outside function f $f(1) = 1^3 + 1 = 2$ ii. $fg(x)$ Inside function is $g(x) = x - 1$ There is nothing to simplify first, since it is terms of x. Put this into the outside function. Don't let the fact that we don't have a number now confuse you. Everywhere we see an x in the outside function we write $x - 1$ $f(x-1) = (x-1)^3 + 1$ iii. $(f \circ f \circ f)(1)$ $f(1) = 1^3 + 1 = 2$ $f(2) = 2^3 + 1 = 9$ $f(9) = 9^3 + 1 = 730$	i. $gh(1)$ Do inside function: $h(1) = \frac{3}{7}$ Put this into outside function $g\left(\frac{3}{7}\right) = \frac{1}{\frac{3}{7}-2} = \frac{7}{3-14} = -\frac{7}{11}$ ii. $hg(x)$ Nothing to do to inside function $g(x) = \frac{1}{x-2}$ $h\left(\frac{1}{x-2}\right)$ $= \frac{2\left(\frac{1}{x-2}\right)+1}{3\left(\frac{1}{x-2}\right)+4}$ $= \frac{2+(x-2)}{3+4(x-2)}$ $= \frac{x}{4x-5}$	$gf(a) = 3$ Let's work out LHS first $f(a) = \frac{2}{a}$ $g\left(\frac{2}{a}\right) = \frac{\frac{2}{a}+1}{\frac{2}{a}}$ Given $gf(a) = 3$ $\frac{\frac{2}{a}+1}{\frac{2}{a}} = 3$ $\Rightarrow a = 4$	Here we have a composite function on both sides Let's work out each side $fg(x) = gf(x)$ LHS: $fg(x) = 2(x^2+2) - 3 = 2x^2 + 1$ RHS: $gf(x) = (2x-3)^2 + 2$ $= 4x^2 - 12x + 11$ Given $fg(x) = gf(x)$ $2x^2 + 1 = 4x^2 - 12x + 11$ $2x^2 - 12x + 10 = 0 \Rightarrow x = 1, 5$

Example 5:	Example 6: Harder Algebra
$g(x) = x^2 + 3, h(x) = 2x + 2$. Solve $gh(x) = 2hg(x) + 15$	$f(x) = \frac{3x-5}{x+1}, x \in \mathbb{R}, x \neq -1$ Show that $ff(x) = \frac{x+a}{x-1}, x \in \mathbb{R}, x \neq -1, x \neq 1$. State the value of a.
$gh(x) = 2hg(x) + 15$ Let's work out LHS first: $gh(x) = (2x+2)^2 + 2$ Now let's work out the RHS: $2hg(x) + 15$ Let's first find the part $h(g(x))$. We will colour code as $hg(x)$ $g(x) = x^2 + 3$ $hg(x) = 2(x^2+3) + 2$ So, $2hg(x) + 15 = 2[2(x^2+3) + 2] + 15$ $gh(x) = 2hg(x) + 15$ becomes $(2x+2)^2 + 2 = 2[2(x^2+3) + 2] + 15$ Simplifying and solving $4x^2 + 8x + 7 = 2(2x^2 + 6 + 2) + 15$ $4x^2 + 8x + 7 = 2(2x^2 + 8) + 15$ $4x^2 + 8x + 7 = 4x^2 + 16 + 15$ $8x = 24$ $x = 3$	We calculate $ff(x)$ the same way we always do. $f(f(x))$ $= \frac{3(f(x)) - 5}{f(x) + 1} = \frac{3\left(\frac{3x-5}{x+1}\right) - 5}{\frac{3x-5}{x+1} + 1}$ **Way 1: Multiply all terms by $x+1$ to kill the fractions quickly** $= \frac{3(3x-5) - 5(x+1)}{3x - 5 + (x+1)}$ $= \frac{9x - 15 - 5x - 5}{4x - 4}$ $= \frac{4x-20}{4x-4}$ $= \frac{x-5}{x-1}$ **Way 2: get a common denominator** $= \frac{\left(\frac{3(3x-5)}{x+1} - \frac{5(x+1)}{x+1}\right)}{\frac{3x-5}{x+1} + \frac{x+1}{x+1}}$ $= \frac{\frac{4x-20}{x+1}}{\frac{4x-4}{x+1}}$ $= \frac{4x-20}{4x-4}$ $= \frac{x-5}{x-1}$ $a = -5$

You should now understand the following memes ☺



Watch out for different notations:

$fg(x)$
$f(g(x))$
$(f \circ g)(x)$
$f \circ g(x)$
$fg: x$

These all mean the same thing

Note: ∘ is **not** a filled in dot, so doesn't mean multiply

Also be aware that

$f^2(x) = f(f(x))$. Don't confuse this with the derivative where $f^{(2)}(x) = f''(x)$

Don't confuse repeated composition and derivatives $f^2(x) = f(f(x))$ not $f''(x)$

An Important Deeper Understanding of Functions

Fact 1: Any function can just be replaced with the letter 'y' :
y is typically used to represent the **output value** of a function. So $f(x)$ can be replaced with y.

$f(x) = x^2$ can also be written as $y = x^2$
$g(x) = x^2$ can also be written as $y = x^2$
$h(x) = x^2$ can also be written as $y = x^2$

Careful though: This does not mean that $f(x)$, $g(x)$ and h(x) are the same functions since they are all equal to y above. It just means both functions are being plotted on the y-axis (hence the y = since y is the axis name), Know that we only do this if finding the inverse or for graphing.

So, going back to our very first example, $f(x) = 3x - 4$ can also be written as $y = 3x - 4$ and then graphed as usual.

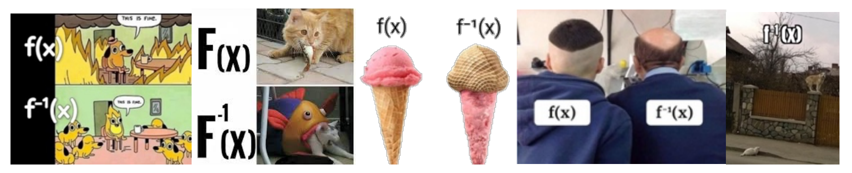
Fact 2: More confusing notations used:
You may also see written in other less common ways using colons and arrows instead of brackets and equals signs. Therefore $f(x) = 3x - 4$ can be written as

$f: x \mapsto 3x - 4$ or $f: x \mapsto y$

Inverses

Inverse Functions – Finding Them

Intution
An inverse swaps things. In the case of functions it says swaps 'x' and 'y'.



Replace function with y. Then swap x and y and then re-arrange to make y the subject

Notation
The notation for the inverse is $f^{-1}(x)$. The **superscript −1** tells us that we are finding the inverse of function f, it **does NOT mean a power of −1**

Method to find an inverse:

- **Step 1:** Replace function with y since $f(x)$ means the same thing as y
- **Step 2:** Swap x and y
- **Step 3:** Make y the subject
- **Step 4:** Replace the y found in step 3 with $f^{-1}(x)$

So, the above steps just answer the question, given a function $f(x)$, how do I find $f^{-1}(x)$?

Note: Some teachers teach you to not to do step 2. Instead they go straight to step 3 and instead make x the subject in step 3 and then swap the letters at the very end. This is also ok.

The difficulty with finding the inverse for most students lies in step 3. In order to make y the subject of a formula, the formula needs to be arranged and hence algebra used. Make sure you're good at the topic algebraic re-arranging/changing the subject!

Example 1	Example 1	Example 3: Two terms with y (factorise)	Example 4: With logarithms
$f(x) = \sqrt{2x-5}$. Find $f^{-1}(x)$	$f(x) = 3x^2 - 5$. Find $f^{-1}(x)$	$f(x) = \frac{2-x}{4+x}$. Find $f^{-1}(x)$ 24	$f(x) = \ln(x+2) + ln2, x \geq -5$ Find $f^{-1}(x)$
Replace the function $f(x)$ with y $y = \sqrt{2x-5}$ Swap x and y $y = \sqrt{2x-5}$ $x = \sqrt{2y-5}$ Make y the subject. We square both sides to get rid of the square root $x^2 = 2y - 5$ $y = \frac{x^2+5}{2}$ $f^{-1}(x) = \frac{x^2+5}{2}$ Note: $f^{-1}(x) = \frac{-x^2-5}{-2}$ is also an acceptable answer	Replace the function $f(x)$ with y $y = 3x^2 - 5$ Swap x and y $y = 3x^2 - 5$ becomes $x = 3y^2 - 5$ Make y the subject. We need to get y^2 on its own before taking the square root $3y^2 = x + 5$ $y^2 = \frac{x+5}{3} \Rightarrow y = \pm\sqrt{\frac{x+5}{3}}$ Note that we need to restrict the domain of this function $y = 3x^2 - 5$ to $x \geq 0$, as this function is not one-to-one (or we could restrict the domain to $x < 0$, and take the negative square root). $f^{-1}(x) = \sqrt{\frac{x+5}{3}}$	Replace function with y $y = \frac{2-x}{4+x}$ Swap x and y $x = \frac{2-y}{4+y}$ Now, make y the subject. We now have two y's. The way to deal with this is to then move all the terms with y's onto one side to be able to factorise y out and hence get y on its own. $x(4+y) = 2 - y$ $4x + xy = 2 - y$ $xy + y = 2 - 4x$ $y(x+1) = 2 - 4x$ $y = \frac{2-4x}{x+1}$ $f^{-1}(x) = \frac{2-4x}{x+1}$ Note: $\frac{4x-2}{-x-1}$ is also an acceptable answer	$f(x) = \ln(x+2) + \ln 2$ $\ln(y+2) + \ln 2 = x$ $\ln(2y+4) = x$ $2y + 4 = e^x$ $y = \frac{e^x - 4}{2}$ $f^{-1}(x) = \frac{e^x-4}{2}$ **Example 5: With exponentials** $f(x) = e^{x+3}$. Find $f^{-1}(x)$ $f(x) = e^{x+3}$ $x = e^{y+3}$ $\ln x = y + 3$ $y = \ln x - 3$ $f^{-1}(x) = \ln x - 3$

Example 6: Quadratic (use quadratic formula or complete the square)

Find the inverse of $f(x) = 1 + x - 2x^2$

We know we have to swap the $x's$ and $y's$ to get $x = 1 + y - 2y^2$, but then how do we re-arrange for y after? We can't factorise y out from all terms like the usual harder types. We have no option but to complete the square or use quadratic formula.

Way 1: Complete the square	Way 2: Use Quadratic Formula (shorte
$x = -2\left(y^2 - \frac{1}{2}y - \frac{1}{2}\right)$ $x = -2\left[\left(y - \frac{1}{4}\right)^2 - \frac{1}{16} - \frac{1}{2}\right]$ $= -2\left[\left(y - \frac{1}{4}\right)^2 - \frac{9}{16}\right]$ $x = -2\left(y - \frac{1}{4}\right)^2 - \frac{9}{8}$ Re-arrange for y $2\left(y - \frac{1}{4}\right)^2 = -x + \frac{9}{8}$ $\left(y - \frac{1}{4}\right)^2 = \frac{-x + \frac{9}{8}}{2}$ $y - \frac{1}{4} = \pm\sqrt{\frac{-x + \frac{9}{8}}{2}}$ $y = \frac{1}{4} \pm \sqrt{\frac{-8x+9}{16}}$ $= \frac{1 \pm \sqrt{-8x+9}}{4}$	Re-arrange first to get 0 on one side $2y^2 - y + x - 1 = 0$ $a = 2, b = -1, c = x - 1$ $y = \frac{1 \pm \sqrt{(-1)^2 - 4(2)(x-1)}}{2(2)}$ $y = \frac{1 \pm \sqrt{-8x+9}}{4}$ $f^{-1}(x) = \frac{1 \pm \sqrt{-8x+9}}{4}$

We need the inverse to exist so there can only be one solution since the function cannot be many to one

$$f^{-1}(x) = \frac{1 + \sqrt{-8x+9}}{4}$$

Unless you limit the domain of $f(x)$ of course (this is covered in the domain and range cheat sheet).
For example,
Domain $\left(-\infty, \frac{1}{4}\right]$ has inverse:

$$f^{-1}(x) = \frac{1 - \sqrt{9-8x}}{4}$$

Domain $\left[\frac{1}{4}, -\infty\right)$ has inverse:

$$f^{-1}(x) = \frac{1+\sqrt{9-8x}}{4}$$

The question will tell you what the domain of x is so you know which to choose (if not just choose the plus case).
We can't have both since then the function will be many to one.
If the question doesn't tell you just use the plus case.

Example 7: Solving with inverse

$g(x) = \frac{4x}{3-x}, f(x) = 2x - 5$. Given that $x > 3$. Solve $g^{-1}(x) = f(x)$

Way 1:	Way 2: Clever way
$g^{-1}(x) = f(x)$ First, let's work out $g^{-1}(x)$, $y = \frac{4x}{3-x}$ $x = \frac{4y}{3-y}$ $x(3-y) = 4y$ $3x - xy = 4y$ $4y + xy = 3x$ $y(4+x) = 3x$ $y = \frac{3x}{4+x}$ $\frac{3x}{4+x} = 2x - 5$ $3x = (2x-5)(4+x)$ $3x = 8x + 2x^2 - 20 - 5x$ $2x^2 - 20 = 0$ $x^2 = 10$ $x = \pm\sqrt{10}$ But question says $x > 3$ $x = \sqrt{10}$	Remembering that $gg^{-1}(x) = x$, as they are inverses. We can apply g to both sides of the equation $g^{-1}(x) = f(x)$. $gg^{-1}(x) = gf(x)$ The inverses on the LHS cancel $x = gf(x)$ Now, let's work out the RHS $gf(x)$. $gf(x) = g(f(x)) = \frac{4(f(x))}{3 - f(x)}$ $= \frac{4(2x-5)}{3-(2x-5)}$ $= \frac{8x-20}{8-2x}$ We substitute it into the equation $x = gf(x)$. $x = \frac{8x-20}{8-2x}$ $x(8-2x) = 8x - 20$ $8x - 2x^2 = 8x - 20$ $-2x^2 = -20$ $x^2 = 10$ $x = \pm\sqrt{10}$ But, question says $x > 3$, $x = \sqrt{10}$

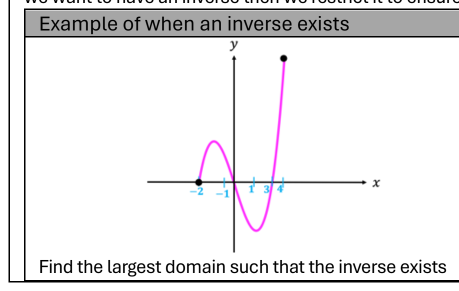
Inverse and composite functions together in one question (typical exam questions)

Example 8:	Example 9: Harder Algebra (2 fractions)	Example 10: Self Inverse
$g(x) = x - 3, f(x) = x^2$. Solve $gf(x) = g^{-1}(x)$	$f(x) = \frac{4}{x-3}, g(x) = \frac{x-2}{x}$. i.Solve $fg(a) = 1$ ii. Find $f^{-1}(x)$	$f(x) = \frac{1-x}{1+x}$ Show that $ff(x) = x$. What does this mean?
$gf(x) = g^{-1}(x)$ $gf(x)) = x^2 - 3$ Now, work out the RHS, $g^{-1}(x)$, using the steps we've practiced before. Set $g(x)$ to y, swap x, y, then make y the subject. $y = x - 3$ $x = y - 3$ $y = x + 3$ $g^{-1}(x) = x + 3$. So $gf(x) = g^{-1}(x)$ becomes $x^2 - 3 = x + 3$ $x^2 - x - 6 = 0$ $(x-3)(x+2) = 0$ $x = 3, -2$	i. $fg(a) = 1$ $g(a) = \frac{a-2}{a}$ $f(g(a)) = \frac{4}{\frac{a-2}{a} - 3}$ $\frac{4}{\frac{a-2}{a} - 3} = 1$ There are 2 ways to proceed from here to get rid of the fractions in order to solve Way 1: Multiply all terms by a on the LHS $\frac{4}{\frac{a-2}{a} - 3} = 1$ $\frac{4a}{\left(\frac{a-2}{a}\right)a - 3a} = 1$ $\frac{4a}{a - 2 - 3a} = 1$ $\frac{4a}{-2a-2} = 1$ $4a = -2a - 2$ $6a = -2$ $a = -\frac{1}{3}$ Way 2: Get a common denominator $\frac{4}{\frac{a-2}{a} - 3} = 1$ $\frac{4}{\frac{a-2}{a} - \frac{3a}{a}} = 1$ $\frac{4}{\frac{a-2-3a}{a}} = 1$ $\frac{4}{\frac{-2a-2}{a}} = 1$ $4 \div \frac{-2a-2}{a} = 1$ $\frac{4}{1} \times \frac{a}{-2a-2} = 1$ $\frac{4a}{-2a-2} = 1$ $4a = -2a - 2$ $a = -\frac{1}{3}$	$ff(x) = \frac{1 - \frac{1-x}{1+x}}{1 + \frac{1-x}{1+x}}$ $= \frac{\frac{1+x}{1+x} - \frac{1-x}{1+x}}{\frac{1+x}{1+x} + \frac{1-x}{1+x}}$ Let's colour code now for ease fo solving $= \frac{\frac{1+x-(1-x)}{1+x}}{\frac{1+x+(1-x)}{1+x}}$ $= \frac{\frac{2x}{1+x}}{\frac{2}{1+x}} = \frac{2x}{1+x} \div \frac{2}{1+x} = \frac{2x}{1+x} \times \frac{1+x}{2} = \frac{2x}{2} = x$ $ff(x) = x$ means that applying f twice gave the original value x, so f is its own inverse Note: If we find that $ff(x) = x$ meaning the $f's$ cancel each other out, we say that f is **self inverse.** Applying f twice gave the original value x, so f is its own inverse. If $ff(x) = x$ we call the function f self inverse

Existence of Inverse

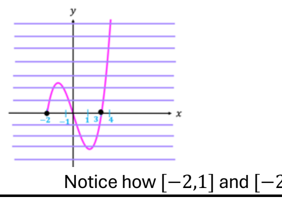
In order to have an inverse a function must not be many one, meaning it must pass the horizontal line test (and of course the vertical). If a function does not and we want to have an inverse then we restrict it to ensure it does.

Example of when an inverse exists



Find the largest domain such that the inverse exists

We want the graph to pass the horizontal test to have an inverse (be one to one not many to one)
The function is one to one on lots of intervals such as



$[-2, -1]$
$[-1, 1]$
$[1, 4]$
$[-1, 0]$

etc
The largest domains is $[1, 4]$
Notice how $[-2, 1]$ and $[-2, 3]$ don't work since the horizontal line would cross twice

Typical Composite and Inverse Past Paper Questions

The following are all typical questions, but do not include domain and range (this is covered in another cheat sheet)

Example 11:	Example 12:	Example 13:
$g(x) = 3 - 4x, x \in \mathbb{R}$. Solve $gg(x) + [g(x)]^2 = 0$	$g(x) = \frac{x}{x+3} + \frac{3(2x+1)}{x^2+x-6}, x > 3$ i.Show that $g(x) = \frac{x+1}{x-2}, x > 3$ ii.Find the exact value of a for which $g(a) = g^{-1}(a)$	$f(x) = e^{2x} + k^2, x \in \mathbb{R}$, k is a positive constant $g(x) = \ln(2x), x > 0$ i.Solve $g(x) + g(x^2) + g(x^3) = 6$. Give your answer in its simplest form ii.Find in terms of the constant k, the solution of the equation $fg(x) = 2k^2$

Example 11:

$gg(x) + [g(x)]^2 = 0$

Remember, $gg(x)$ means apply g twice, and $[g(x)]^2$ means squaring the result of applying g.

Work out $gg(x)$:

$$gg(x) = g(g(x)) = 3 - 4(g(x)) = 3 - 4(3 - 4x) = 3 - 12 + 16x = 16x - 9$$

Now, work out $[g(x)]^2$:

$$g(x)^2 = (3 - 4x)^2 = 16x^2 - 24x + 9$$

Let's plug these in:

$$16x - 9 + 16x^2 - 24x + 9 = 0$$
$$16x^2 - 8x = 0$$
$$2x^2 - x = 0$$
$$x(2x - 1) = 0$$
$$x = 0 \text{ or } \frac{1}{2}$$

Example 12:

i.

$$\frac{x}{x+3} + \frac{3(2x+1)}{x^2+x-6} = \frac{x}{x+3} + \frac{3(2x+1)}{(x+3)(x-2)}$$
$$\frac{x(x-2)}{(x+3)(x-2)} + \frac{3(2x+1)}{(x+3)(x-2)}$$
$$= \frac{x^2+4x+3}{(x+3)(x-2)} = \frac{(x+3)(x+1)}{(x+3)(x-2)} = \frac{x+1}{x-2}$$

ii.

$$g(a) = g^{-1}(a)$$

First find $g^{-1}(a)$

$$y = \frac{x+1}{x-2}$$
$$x = \frac{y+1}{y-2}$$
$$(y-2)x = y+1$$
$$yx - 2x = y + 1$$
$$yx - y = 1 + 2x$$
$$y(x-1) = 1 + 2x$$
$$y = \frac{1+2x}{x-1}$$

$g^{-1}(x) = \frac{1+2x}{x-1}$ hence $g^{-1}(a) = \frac{1+2a}{a-1}$

$g(a) = \frac{a+1}{a-2}$.

So $g(a) = g^{-1}(a)$ becomes,

$$\frac{a+1}{a-2} = \frac{1+2a}{a-1}$$

Get rid of the fractions by cross multiplying

$$(a-1)(a+1) = (1+2a)(a-2)$$

Expand the brackets

$$a^2 - 1 = 2a^2 - 3a - 2$$
$$a^2 - 3a - 1 = 0$$

Apply the quadratic formula, we get.

$$a = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-1)}}{2(1)} = \frac{3 \pm \sqrt{13}}{2}$$

Remember that the domain of g requires that $x > 3$, so $a > 3$.

$$a = \frac{3 + \sqrt{13}}{2}$$

Example 13:

i.

Let's expand the corresponding terms of

$$g(x) + g(x^2) + g(x^3) = 6$$
$$\ln(2x) + \ln(2x^2) + \ln(2x^3) = 6$$

the log law that $\ln(a) + \ln(b) = \ln(ab)$

$$\ln(2x \times 2x^2 \times 2x^3) = 6$$
$$\ln(8x^6) = 6$$
$$8x^6 = e^6$$
$$x^6 = \frac{e^6}{8}$$
$$x = \left(\frac{e^6}{8}\right)^{\frac{1}{6}} = \frac{e}{8^{\left(\frac{1}{6}\right)}} = \frac{e}{(2^3)^{\frac{1}{6}}} = \frac{e}{\sqrt{2}}$$

We only need to take the positive root because we know $x > 0$.

ii.

First, we need to calculate the LHS, $fg(x)$.

$$fg(x) = f(g(x)) = e^{2(g(x))} + k^2 = e^{2\ln(2x)} + k^2$$
$$= \left(e^{\ln(2x)}\right)^2 + k^2$$
$$= (2x)^2 + k^2$$
$$= 4x^2 + k^2$$

Now, equate LHS and RHS.

$$4x^2 + k^2 = 2k^2$$
$$4x^2 = k^2$$
$$x^2 = \frac{k^2}{4}$$
$$x = \pm\frac{k}{2}$$

But we know $x > 0$, from the given domain of g.

$$x = \frac{k}{2}$$

Example 14: Hard	Example 15: Very Hard	Example 16: With logs
$f(x) = 7x - 1, g(x) = \frac{4}{x-2}, x \neq 2, x \in \mathbb{R}$ i.Solve the equation $fg(x) = x$ ii.Hence, or otherwise, find the **largest value** of a such that $g(a) = f^{-1}(a)$	$g(x) = 3 + \sqrt{x+2}, x \geq -2$ i.Find $g^{-1}(x)$ ii.Find the exact value of x for which $g(x) = x$ iii.Hence state the value of a for which $g(a) = g^{-1}(a)$	$f(x) = k\log_2 x$, Given that $f^{-1}(1) = 8$ i.find k find $f^{-1}\left(\frac{2}{3}\right)$

Example 14:

i.

Work out the LHS:

$$fg(x) = f(g(x)) = 7(g(x)) - 1$$
$$= 7\left(\frac{4}{x-2}\right) - 1$$
$$= \frac{28}{x-2} - 1$$
$$= \frac{28}{x-2} - \frac{x-2}{x-2}$$
$$= \frac{30-x}{x-2}$$

Now, equate both sides of the equation.

$$\frac{30-x}{x-2} = x$$
$$30 - x = x(x-2)$$
$$30 - x = x^2 - 2x$$
$$x^2 - x - 30 = 0$$
$$(x-6)(x+5) = 0$$
$$x = 6 \text{ or } -5$$

ii.

If we apply f to both sides of the equation of $g(a) = f^{-1}(a)$. We get $fg(a) = ff^{-1}(a) = a$.

So, the values of a that satisfy the equation $g(a) = f^{-1}(a)$ is precisely the possible values of x that satisfy $fg(x) = x$. This is the exact same as i. From the previous part, x is either 6 or -5. The largest value is therefore 6.

Example 15:

i.

$$x = 3 + \sqrt{y+2}$$
$$\sqrt{y+2} = x - 3$$
$$y = (x-3)^2 + 2$$
$$g^{-1}(x) = (x-3)^2 + 2$$

ii.

$$g(x) = x$$

We replace $g(x)$ with $3 + \sqrt{x+2}$.

$$3 + \sqrt{x+2} = x$$
$$\sqrt{x+2} = x - 3$$
$$x + 2 = (x-3)^2$$
$$x + 2 = x^2 - 6x + 9$$
$$x^2 - 7x + 7 = 0$$

Using the quadratic formula

$$x = \frac{7 \pm \sqrt{21}}{2}$$

$x = \frac{7+\sqrt{21}}{2}$ is the only solution since domain of $g^{-1}(x)$ is $x \geq 3$ and $\frac{7-\sqrt{21}}{2} < 3$

(see domain cheat sheet for how)

Alternatively, to see which solution to ignore, we can check for these extraneous solutions.

When squaring both sides of the equation earlier to get rid of the square root we needed to take care, as this can introduce extra solutions! (eg $x = 1$ has a single solution, $x = 1$, but squaring both sides to get $x^2 = 1$, gives $x = \pm 1$). So, let's check for these extraneous solutions

To do this we plug the values found back into the original equation $3 + \sqrt{x+2} = x$ and check if it's true.

$$g\left(\frac{7+\sqrt{21}}{2}\right) = 3 + \sqrt{\frac{7+\sqrt{21}}{2} + 2} = \frac{7+\sqrt{21}}{2}$$

$$g\left(\frac{7-\sqrt{21}}{2}\right) = 3 + \sqrt{\frac{7-\sqrt{21}}{2} + 2} \neq \frac{7-\sqrt{21}}{2}$$

$\frac{7-\sqrt{21}}{2}$ doesn't satisfy the original equation, so $\frac{7+\sqrt{21}}{2}$ is the only solution

iii.

Clearly, if $g\left(\frac{7+\sqrt{21}}{2}\right) = \frac{7+\sqrt{21}}{2}$, then re-arranging gives $\frac{7+\sqrt{21}}{2} = g^{-1}\left(\frac{7+\sqrt{21}}{2}\right)$ as well.

So $a = \frac{7+\sqrt{21}}{2}$ is a solution.

Example 16:

i.

$$k\log_2 y = x$$
$$\log_2 y = \frac{x}{k}$$
$$y = 2^{\frac{x}{k}}$$
$$f^{-1}(x) = 2^{\frac{x}{k}}$$
$$f^{-1}(1) = 2^{\frac{1}{k}} = 8$$
$$2^{\frac{1}{k}} = 2^3$$
$$\frac{1}{k} = 3$$
$$k = \frac{1}{3}$$

ii.

$$f^{-1}(x) = 2^{\frac{x}{\frac{1}{3}}} = 2^{3x}$$
$$f^{-1}\left(\frac{2}{3}\right) = 2^{3\left(\frac{2}{3}\right)} = 2^2 = 4$$

Example 17 : with 3 compositions

$f(x) = 2x + 2$
$g(x) = x^2$
$h(x) = \frac{1}{x}$

Find $f\left(g(h(x))\right)$

$f\left(g(h(x))\right)$

Apply h
Apply g
Apply f

$$h(x) = \frac{1}{x}$$
$$g(h(x)) = \left(\frac{1}{x}\right)^2 = \frac{1}{x^2}$$
$$f\left(g(h(x))\right) = 2\left(\frac{1}{x^2}\right) + 2 = \frac{2}{x^2} + 2$$

Example 17 : Reading off data tables

Find $f\left(g(h(2))\right)$

x	f	g	h
0	5	2	−1
1	4	−2	3
2	9	1	4
3	1	0	5
4	3	2	1

x	f	g	h
0	5	2	−1
1	4	−2	3
2	9	1	4
3	1	0	5
4	3	2	1

$f\left(g(h(2))\right)$

$h(2) = 4$

$g(4) = 2$

$f(2) = 9$

Operations With Functions $(+, -, \times, \div)$

We don't just plug values into functions. We can also add, subtract, multiply and divide functions. The result is a new function.

Tip: when solving the equation where a function equals its inverse i.e. $f(x) = f^{-1}(x)$ we can instead solve $f(x) = x$ since the points where a function intersects its inverse always lie on the line $y = x$. This is an important concept and saves a lot of time.

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

$$(f + g)(x) = f(x) + g(x)$$

So, addition and division together would look like

$$\left(\frac{f+g}{h}\right)(x) = \frac{f(x)+g(x)}{h(x)}$$

This **never ever comes up**, but you should understand it anyway **just in case**. The best way to understand this is with examples.

Example 1

$f(x) = 3x - 8$ and $g(x) = 3x^2 - 5x - 8$

Find

i. Find $(f+g)(x)$
ii. Find $(g-f)(x)$
iii. Find $(f \cdot g)(x)$
iv. Find $\left(\frac{f}{g}\right)(x)$

i.

$(f+g)(x)$

Here we add the functions together.

$$(3x - 8) + (3x^2 - 5x - 8)$$
$$= 3x^2 - 2x - 16$$

ii.

$(g-f)(x)$

Here we subtract the functions.

$$(3x^2 - 5x - 8) - (3x - 8)$$
$$= 3x^2 - 8x$$

iii.

$(f \cdot g)(x)$

Here we multiply the functions together (this is a not an empty circle like for composition)

$$(3x - 8)(3x^2 - 5x - 8)$$
$$= 9x^3 - 39x^2 + 16x + 64$$

iv.

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

Here we divide the functions.

$$\frac{3x - 8}{3x^2 - 5x - 8}$$
$$= \frac{3x - 8}{(3x - 8)(x + 1)}$$
$$= \frac{1}{x + 1}$$

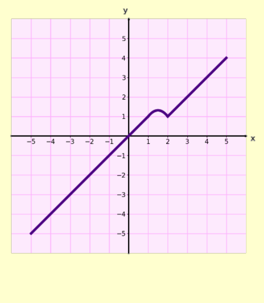
Graphing With Function Notation

Recall $f(x)$ is the same as saying the y value of the axis on a graph. $f(x) = y$ means a function takes an input of x and returns an output y.

$f(3) = 2$ (Tells us $x = 3$, Tells us $y = 2$) $\quad f^{-1}(3) = 2$ (Tells us $y = 3$, Tells us $x = 2$)

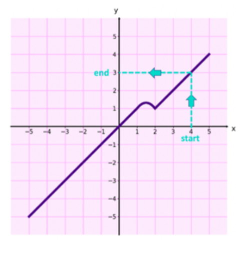
Example 1: Basics

Let's look at a graphical example. Given the graph below



Use the graph to find the following
i. $f(4)$
ii. $f(-2)$
iii. Solve $f(x) = 2$

i.
We look at the where the point on the graph where $x = 4$, and find the corresponding value of y



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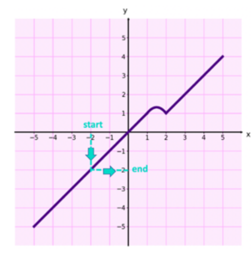
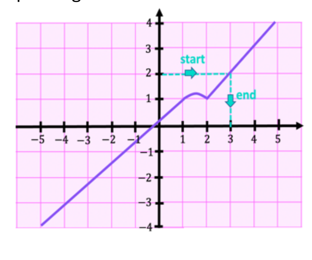
ii.
We look at the where the point on the graph where $x = -2$, and find the corresponding value of y



−2

iii.
We look at the where the point on the graph where $y = 2$, and find the corresponding value of x value



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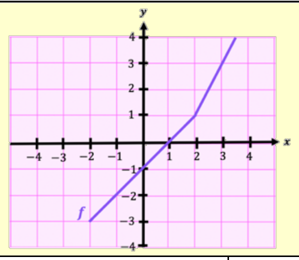
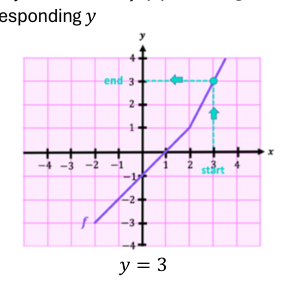
Example 2: With Composite

Consider the following graph $f(x)$
i.Write down the value of $f(3)$
ii.Write down the value of $ff(0)$



i.
Recall that $f(x) = y$. This means $f(3)$ is telling us $x = 3$ and wants us to find the corresponding y

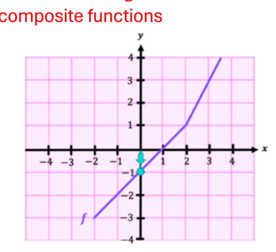


$y = 3$

So $f(3) = 3$

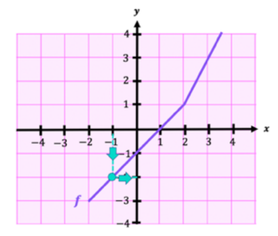
ii.
$ff(0)$

Step 1:
We work from right to left with composite functions



$f(0) = -1$

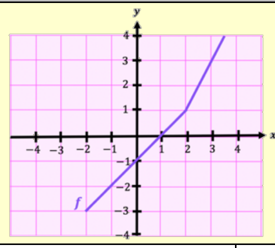
Step 2:
Now we want to find $f(-1)$



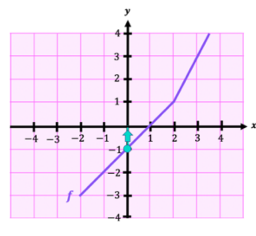
$f(-1) = -2$

Example 3 – With Inverse

Consider the following graph $f(x)$
i.Write down the value of $f^{-1}(-1)$
ii.Sketch the inverse $f^{-1}(x)$



i.
Recall that $f^{-1}(y) = x$
Here is telling us $y = -1$ and wants us to find x.

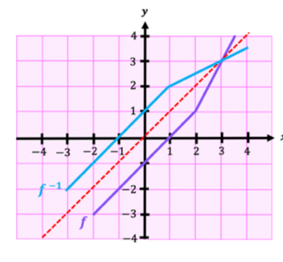


Here we can see that when $y = -1$ we get $x = 0$

0

ii.
We swap the x and y coordinates which gives us the points:
$(-2,-3), (-1,-2), (0,-1), (1,0), (2,1), (3,3)$
Become. $(-3,-2), (-2,-1), (-1,0), (0,1), (1,2), (3,3)$

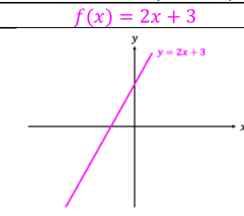
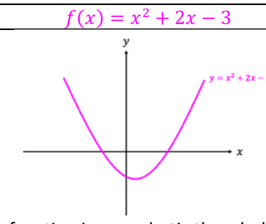
Now plot these points



Notice how an inverse (f^{-1}) is just a reflection of the original graph (f) in the line $y = x$ hence the points of each coordinates are swapped

Piecewise Functions

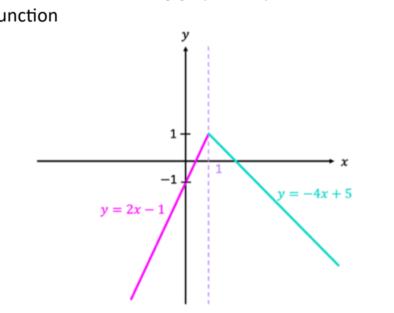
So far you have seen functions that are made up of one piece. For example

$f(x) = 2x + 3$	$f(x) = x^2 + 2x - 3$
	
The function is a straight line with positive slope the whole time.	The function is a quadratic the whole time.

A piecewise function is a function that is in pieces (it is not given by the same equation the whole time). This means that we create functions that behave differently based on the input (x) value. In other words, the functions behave differently for different values of x.

Example 1:

Consider the following graph of a piecewise function



Find $f(-1)$ and $f(5)$ like you are familiar with functions.

Answer

This function is made up of 2 pieces (the 2 different colours) and they do different things for different values of x (the purple dashed line is not part of the graph it was just drawn to demonstrate/highlight the fact that the function looks different for values of x less than 1 than what it does for values of x greater than 1).

This is a piecewise function as it consists of 2 different functions:
- **Function 1:** a straight diagonal line with **positive** slope for x values less than or equal to 1
- **Function 2:** a straight diagonal line with **negative** slope for x values greater than 1

We can write this function with its equation as

$$f(x) = \begin{cases} 2x - 1, & x \leq 1 \\ -4x + 5, & x > 1 \end{cases}$$

$f(-1)$:

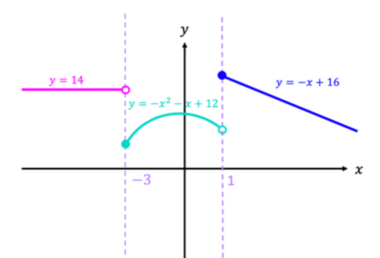
When $x = -1$ the function $f(x)$ looks like $2x - 1$
We plug in $x = -1$ into this function
$f(-1) = 2(-1) - 1 = -3$

$f(5)$:

When $x = 5$ the function looks like $-4x + 5$
We plug in $x = 5$ into this function
$f(5) = -4(5) + 5 = -15$

Example 2:

Consider the following graph of a piecewise function



Find $f(0)$, $f(5)$, and $f(-4)$

Answer

This function made up of 3 pieces (the 3 different colours) and they do different things for different values of x
- **Function 1:** a straight horizontal line for x values less than -3
- **Function 2:** a curve for x values between -3 and 1
- **Function 3:** a straight diagonal line for x values greater than or equal to 1

We can write this function with its equation as

$$f(x) = \begin{cases} 14, & x < -3 \\ -x^2 - x + 12, & -3 \leq x < 1 \\ -x + 16, & x \geq 1 \end{cases}$$

$f(0)$	$f(5)$	$f(-4)$
When $x = 0$ the function $f(x)$ looks like $-x^2 - x + 12$ We plug in $x = 0$ into this function $f(0) = -0^2 - 0 + 12 = 12$	When $x = 5$ the function $f(x)$ looks like $-x + 16$ We plug in $x = 5$ into this function $f(5) = -5 + 16 = 11$	When $x = -4$ the function $f(x)$ is 14 We plug in $x = -4$ into this function $f(-4) = 14$ Note: Notice how the function didn't depend on x here, it is always 14

Example 3

$$f(x) = \begin{cases} 5, & x \leq 2 \\ 2x - 3, & x > 2 \end{cases}$$

Find $f(-3), f(2), f(3)$

$$f(x) = \begin{cases} 5, & x \leq 2 \\ 2x - 3, & x > 2 \end{cases}$$

$f(-3)$	$f(2)$	$f(3)$
Since $-3 \leq 2$, we substitute -3 for x inside the function 5 This expression is always 5 regardless of the value of x $f(-3) = 5$	Since $2 \leq 2$, we substitute 0 for x inside the function 5 This expression is always 5 regardless of the value of x $f(2) = 5$	Since $3 > 2$, so we substitute 3 for x inside function $2x - 3$. $f(3) = 2(3) - 3 = 3$

Example 4

$$f(x) = \begin{cases} x^2 - 3x, & x \leq 1 \\ 2x - 6, & x > 1 \end{cases}$$

Solve $f(x) = 4$

$$f(x) = \begin{cases} x^2 - 3x, & x \leq 1 \\ 2x - 6, & x > 1 \end{cases}$$

We have to set both of the function equal to 4 and then check whether the answers are in the correct domain/region for the function chosen.

$$x^2 - 3x = 4$$
$$x^2 - 3x - 4 = 0$$
$$(x + 1)(x - 4) = 0$$
$$x = -1, x = 4$$

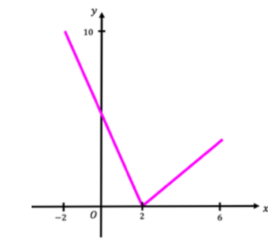
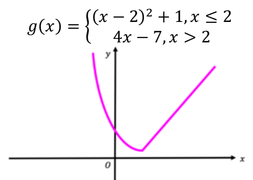
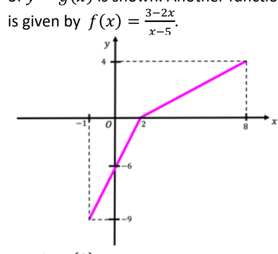
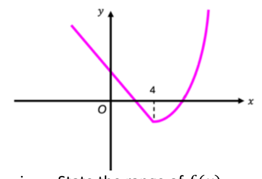
$x = 4$ is not within the domain $x \leq 1$ so we only have $x = -1$

$$2x - 6 = 4$$
$$2x = 10$$
$$x = 5$$

This is within the domain $x > 1$

$x = -1, x = 5$

Piecewise Functions Continued

C3 June 2013 Q7	June 2019 Pure Paper 2 Q6	C3 Jan 2011	C3 June 2013 replaced paper Q6
C3 June 2013 Q7 The function f has domain $-2 \leq x \leq 6$ and is linear from $(-2,10)$ to $(2,0)$ and from $(2,0)$ to $(6,4)$. A sketch of the graph of $y = f(x)$ is shown.  i.Write down the range of f i.Find $ff(0)$ The function g is defined by $g(x) = \frac{4+3x}{5-x}$, $x \in \mathbb{R}, x \neq 5$ ii.Find $g^{-1}(x)$ iii.Solve $gf(x) = 16$	**June 2019 Pure Paper 2 Q6** The diagram shows a sketch of the graph $y = g(x)$ where $g(x) = \begin{cases} (x-2)^2 + 1, x \leq 2 \\ 4x - 7, x > 2 \end{cases}$  i.Find the value of $gg(0)$ ii.Find all values of x for which $g(x) > 28$ The function h is defined by $h(x) = (x-2)^2 + 1, x \leq 2$ iii.Explain why h has an inverse but g does not iv.Solve the equation $h^{-1}(x) = -\frac{1}{2}$	The function g has domain $-1 \leq x \leq 8$ and is linear from $(-1,9)$ to $(2,0)$ and from $(2,0)$ to $(8,4)$. A sketch of the graph of $y = g(x)$ is shown. Another function f is given by $f(x) = \frac{3-2x}{x-5}$.  i.Find $gg(2)$ ii.Find $fg(8)$ iii.Sketch $g^{-1}(x)$	The diagram shows a sketch of the graph $y = f(x)$ where $f(x) = \begin{cases} 5 - 2x, x \leq 4 \\ e^{2x-8} - 4, x > 4 \end{cases}$  i. State the range of $f(x)$ ii. Determine the exact value of $ff(0)$ iii. Solve $f(x) = 21$. Give each answer as an exact answer. iv. Explain why the function f does not have an inverse
Answer i. 3 **ii.**$g^{-1}(x) = \frac{5x-4}{x+3}$ iii.$x = 0.4, x = 6$	**Answer** i. 13 **ii.**$x < 2 - 3\sqrt{3}, x > \frac{35}{4}$ iii.h is one to one and g is many to one iii.$x = 7.25$	**Answer** i. -6 **ii.**5 iii. $(-1,-9) \to (-9,-1)$ $(0,-6) \to (-6,0)$ $(2,0) \to (0,2)$ $(8,4) \to (4,8)$	**Answer** i. $y \geq -3$ ii.$e^2 - 4$ iii.$x = \ln 5 + 4$ **iv.** f has no inverse as is many to one (not one to one)

Hardest Types Of Questions

Inverse Functions and understanding notations

Recall, the notation for the inverse is $f^{-1}(x)$. Also recall that a function is always equal to y. This means just in the same way we can write $y = f(x)$ we can also write $y = f^{-1}(x)$.

We can also re-arrange these by taking the inverse of the other side or undoing the inverse on the other side.

$$y = f(x) \Rightarrow f^{-1}(y) = x$$
or
$$y = f^{-1}(x) \Rightarrow f(y) = x$$

For example,

$$f^{-1}(2) = 5$$

Tells us that

$x = 5$ and $y = 2$ in the function $f(x)$

It also tells us that

$x = 2$, and $y = 5$ in the inverse function $f^{-1}(x)$

Example 1:

Given that $f(x) = k(2 + x)$, find the value of k if $f^{-1}(-2) = -3$

Way 1: without finding the inverse (shorter method)

$$f(x) = k(2 + x)$$

This means the same as

$$y = k(2 + x)$$

Here we are given $f^{-1}(-2) = 3$
This tells us that $y = -2$ and $x = -3$ since it is the inverse and tells us the opposite to $y = f(x)$

Sub $y = -2$ and $x = -3$ into $y = k(2 + x)$:

$$-2 = k(2 - 3)$$
$$-2 = -k$$
$$k = 2$$

Way 2: If finding the inverse

$$f(x) = k(2 + x)$$

This means the same as

$$y = k(2 + x)$$

Find the inverse first by swapping x and y are re-arranging for y, so we get

$$x = k(2 + y)$$
$$x = 2k + ky$$
$$y = \frac{x - 2k}{k}$$
$$f^{-1}(x) = \frac{x - 2k}{k}$$

Here we are given $f^{-1}(-2) = -3$
Sub $x = -2$ and $y = -3$ into the inverse

$$-3 = \frac{-2 - 2k}{k}$$
$$-3k = -2 - 2k$$
$$k = 2$$

Example 2

$f(x) = 2x + 1, g(x) = 5x - 3$

I. $(f \circ g^{-1})(x)$
II. $(f^{-1} \circ g)(x)$
III. $(f^{-1} \circ g^{-1})(x)$
IV. $(f \circ g)^{-1}(x)$

First, let's work out f^{-1} and g^{-1}, as we will need them later.

We do so by setting the function to y, swapping x, y, then making y the subject.

$f^{-1}(x)$	$g^{-1}(x)$
$y = 2x + 1$ $x = 2y + 1$ $2y = x - 1$ $y = \frac{x-1}{2}$ $f^{-1}(x) = \frac{x-1}{2}$	$y = 5x - 3$ $x = 5y - 3$ $5y = x + 3$ $y = \frac{x+3}{5}$ $g^{-1}(x) = \frac{x+3}{5}$

i. $(f \circ g^{-1})(x)$

This says find $g^{-1}(x)$ and then put it into $f(x)$

This is just $fg^{-1}(x)$, we compose the two functions.

$$fg^{-1}(x) = f(g^{-1}(x)) = 2(g^{-1}(x)) + 1 = 2\left(\frac{x+3}{5}\right) + 1 = \frac{2(x+3)+5}{5} = \frac{2x+11}{5}$$

ii. $(f^{-1} \circ g)(x)$

Find $g(x)$ and then put it into $f^{-1}(x)$

This is just $f^{-1}g(x)$, we compose the two functions.

$$f^{-1}g(x) = f^{-1}(g(x)) = \frac{g(x) - 1}{2} = \frac{(5x-3) - 1}{2} = \frac{5x-4}{2}$$

iii. $(f^{-1} \circ g^{-1})(x)$

Find $g^{-1}(x)$ and then put it into $f^{-1}(x)$

Again, we work out $f^{-1}g^{-1}(x)$.

$$f^{-1}g^{-1}(x) = f^{-1}(g^{-1}(x)) = \frac{g^{-1}(x) - 1}{2} = \frac{\frac{x+3}{5} - 1}{2} = \frac{\frac{x-2}{5}}{2} = \frac{x-2}{10}$$

iv. $(f \circ g)^{-1}(x)$

Find $(f \circ g)(x)$ and then find the inverse of that

There are two ways of doing this. Perhaps the more straightforward way is to simply invert $(f \circ g)(x)$, which is just $fg(x)$.

$$fg(x) = f(g(x)) = 2(g(x)) + 1 = 2(5x - 3) + 1 = 10x - 5$$

Now let's invert it.

$$y = 10x - 5$$

Swap x and y.

$$x = 10y - 5$$

Make y the subject.

$$10y = x + 5$$
$$y = \frac{x+5}{10}$$
$$(f \circ g)^{-1}(x) = \frac{x+5}{10}$$

Alternatively, we can reason that to invert $f \circ g$, we need to invert f first, as it is applied last, then we invert g. So $(f \circ g)^{-1}(x) = (g^{-1} \circ f^{-1})(x)$. The result of the composition should give the same answer.

Composite Functions – working backwards (never comes up but included just incase ☺)

Sometimes you're given the composite function result and one of the 2 functions and have to work backwards to find the other function. Not on your course though!

Asked to find inside function:	Asked to find outside function:
$f(x) = x^3, (f \circ g)(x) = 2x + 3$, find $g(x)$	$f(x) = x^3, (g \circ f)(x) = 2x + 3$, find $g(x)$
$(f \circ g)(x) = 2x + 3$ means $f(g(x)) = 2x + 3$ Replace $g(x)$ with y $f(y) = 2x + 3$ Replace $f(y)$ on LHS with what f means from $f(x) = x^3$ $y^3 = 2x + 3$ $y = (2x + 3)^{\frac{1}{3}} = g(x)$	Find $g(x^3) = 2x + 3$ Let $z = x^{\frac{1}{3}} \Leftrightarrow x = z^{\frac{1}{3}}$ so $g(z) = 2x + 3$ We want $g(z)$ in terms of z, not x $g(z) = 2z^{\frac{1}{3}} + 3$ Want x so relabel with $x \Rightarrow g(x) = 2x^{\frac{1}{3}} + 3 = 2\sqrt[3]{x}+3$

Modulus

This is a big topic. See modulus cheat sheet

Graphing Functions

This is a big topic. See how ot graph cheat sheet. It covers all types of functions (quadratic, cubic, logarithmic, exponential, quartic, root, trig, piecewise functions and modulus functions)

Transformations

This is covered in year 1. See transformations cheat sheet

Domain and Range

This is a big topic.
For basic types, composite, inverse and compositive combined with inverse see domain and range cheat sheet
For parametric domain and range see parametric cheat sheet